

Real Forms of the Oscillator Quantum Algebra and its Representations

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Abstract

We consider the conditions under which the q -oscillator algebra becomes a Hopf $*$ -algebra. In particular, we show that there are at least two real forms associated with the algebra. Furthermore, through the representations, it is shown that they are related to $\text{su}_{q^{1/2}}(2)$ with different conjugations.

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The recent focus on a class of algebras called quantum algebras, or sometimes loosely called quantum groups, can be attributed to the fact that it appears to be the basic algebraic structure underlying many of the physical theories of current interest. Indeed, since its emergence from the study of the inverse scattering problem [1], it is now well established that it is also deeply rooted in other areas such as exactly soluble models [2], factorizable S -matrix theory [3] and conformal field theory [4]. Mathematically, it has been shown that these structures can best be described by a general class of associative algebras called Hopf algebras [5].

To date many explicit examples of these algebras have been furnished. In particular, the quantum algebra $\text{su}_q(2)$, by virtue of its simplicity and relevance to physical systems, has attracted a lot of attention. In fact it has now, more or less, assumed a paradigmatic role in the study of quantum algebras. Of no less importance are the oscillator algebras, or in this case q -oscillators. First introduced by Macfarlane [6] and independently by Biedernharn [7], they have been used in giving a realization of $\text{su}_q(2)$.

More recently it has been shown [8] (see also [9]) that the q -oscillator algebra, when expressed in a commutator form, may itself support a Hopf structure. In other words, it itself is a quantum algebra. In the following we examine this algebra under the conditions of a Hopf $*$ -algebra [10-12]. This essentially provides the *real forms* of the algebra or alternatively allows for the restriction of the algebra over the real number field since in physics one is often interested in operators that are hermitian. Here, we show that for the conjugation as implied in the algebra of ref.[8], the only admissible values of q are $\{q \in \mathbf{C}; |q|=1\}$. We also present an alternative conjugation which allows for real q . In both cases, it is shown, through their representations that they are closely related to $\text{su}_{q^{1/2}}(2)$.

We begin by considering the associative algebra $\mathcal{H}_q(1)$ generated by the elements a , \bar{a} and N satisfying [8]

$$[a, \bar{a}] = [N + 1]_q - [N]_q \tag{1a}$$

$$[N, \bar{a}] = \bar{a} \quad [N, a] = -a \tag{1b}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

It should be noted that we differ from the notation of ref.[8] in two respects. Firstly, the right hand side of (1a) as given there is $[N + \frac{1}{2}]_q - [N - \frac{1}{2}]_q$ instead of the one given above. Here we have made the replacement $N \rightarrow N + \frac{1}{2}$. Secondly, \bar{a} instead of a^+ has been used,

as we will also be considering the situation in which a and \bar{a} are not conjugates of one another. The Hopf structure associated with $\mathcal{H}_q(1)$ is given by

$$\Delta(a) = a \otimes q^{1/2(N+\gamma)} + q^{-1/2(N+\gamma)} \otimes a \quad (3a)$$

$$\Delta(\bar{a}) = \bar{a} \otimes q^{1/2(N+\gamma)} + q^{-1/2(N+\gamma)} \otimes \bar{a} \quad (3b)$$

$$\Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1} \quad (3c)$$

$$\epsilon(a) = \epsilon(\bar{a}) = 0 \quad \epsilon(N) = -\gamma \quad (3d)$$

$$S(a) = -q^{-1/2}a \quad S(\bar{a}) = -q^{1/2}\bar{a} \quad S(N) = -N - 2\gamma \mathbf{1} \quad (3e)$$

where, with $q = e^\epsilon$,

$$\gamma = \frac{1}{2} - i \frac{(2l+1)\pi}{2\epsilon} \quad l \in \mathbf{Z}. \quad (4)$$

Here Δ , ϵ and S are the coproduct, counit and antipode respectively.

Next, let us consider its conjugations. In the context of Hopf algebras, this requires the notion of a $*$ -algebra which is also compatible with the underlying Hopf structure. More precisely, one defines a Hopf $*$ -algebra as a Hopf algebra \mathcal{A} which is equipped with an involution $a \mapsto a^+$ of \mathcal{A} into itself such that

$$(\alpha a + \beta b)^+ = \bar{\alpha}a^+ + \bar{\beta}b^+ \quad (ab)^+ = b^+a^+ \quad (a^+)^+ = a \quad (5a)$$

$$\Delta(a^+) = \Delta(a)^+ \quad \epsilon(a^+) = \overline{\epsilon(a)} \quad S(S(a)^+)^+ = a \quad (5b)$$

for any $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbf{C}$ [10]. It should be noted that the above involution is sometimes referred to as the *standard conjugation*. It is also possible to introduce yet another involution, called the *non-standard conjugation*. Here the action on Δ reads as [11,12]

$$\Delta(a)^+ = \Delta'(a^+) \quad (6a)$$

where Δ' is the opposite coproduct while the antipode satisfies

$$S(a^+) = S(a)^+. \quad (6b)$$

Now returning to $\mathcal{H}_q(1)$, let us see whether the canonical conjugation

$$(a)^+ = \bar{a} \quad (\bar{a})^+ = a \quad (N)^+ = N \quad (7)$$

furnished in ref.[8], satisfies the necessary requirements. First as a $*$ -algebra, (5a) requires that the conjugation be compatible with the commutation relations. Explicitly, by conjugating both sides of (1a) and using (7), one has

$$[a, \bar{a}] = [N+1]_{\bar{q}} - [N]_{\bar{q}} \quad (8)$$

where \bar{q} denotes the complex conjugate of q . It is clear then, that for the relations to be preserved we must restrict q to be either real or $|q| = 1$ for $q \in \mathbf{C}$. As for the compatibility with the Hopf structure, one finds that only the $|q| = 1$ case is admissible. To see that it fails for $q \in \mathbf{R}$, it suffices to consider conditions (5b) or (6a) when applied to N . With $N^+ = N$, we have

$$\Delta(N^+) = \Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1} \quad (9a)$$

while

$$\begin{aligned} \Delta(N)^+ &= (N \otimes \mathbf{1} + \mathbf{1} \otimes N + \gamma \mathbf{1} \otimes \mathbf{1})^+ \\ &= N \otimes \mathbf{1} + \mathbf{1} \otimes N + \bar{\gamma} \mathbf{1} \otimes \mathbf{1} \end{aligned} \quad (9b)$$

from which we surmise that $\Delta(N^+) \neq \Delta(N)^+$ since $\gamma \neq \bar{\gamma}$. It is also easy to see that the non-standard conjugation does not hold either. For the $|q| = 1$ case, we note that γ (eqn.(4) with $\epsilon \rightarrow i\epsilon$) now reads as

$$\gamma = \frac{1}{2} - \frac{(2l+1)\pi}{2\epsilon} \quad l \in \mathbf{Z} \quad (10)$$

and is real. In this case both the standard as well as the non-standard conjugation holds for N . For a , however, only the non-standard conjugation is admissible. This is evident from the following:

$$\Delta(a^+) = a^+ \otimes q^{1/2(N+\gamma)} + q^{-1/2(N+\gamma)} \otimes a^+ \quad (11a)$$

$$\Delta'(a^+) = a^+ \otimes q^{-1/2(N+\gamma)} + q^{1/2(N+\gamma)} \otimes a^+ \quad (11b)$$

$$\Delta(a)^+ = a^+ \otimes q^{-1/2(N+\gamma)} + q^{1/2(N+\gamma)} \otimes a^+. \quad (11c)$$

In addition, the conditions on the counit and the antipode are also satisfied by all the generators. Thus we are left to conclude $\mathcal{H}_q(1)$ under the involution (7) is a *bonafide* Hopf $*$ -algebra only for $|q| = 1$. This naturally raises the question as to whether other conjugations exist for which $q \in \mathbf{R}$ is admissible. In other words, are there other real forms associated with $\mathcal{H}_q(1)$.

To this end we generalize (7) and write instead,

$$(a)^+ = \alpha \bar{a} \quad (\bar{a})^+ = \beta a \quad (N)^+ = N + \eta \quad (12)$$

where α, β and η are constants to be determined.[†] By assuming that q is real, it is easy to see that the Hopf $*$ -algebra requirements allow for the following involutions:

$$(a)^+ = \pm i\bar{a} \quad (\bar{a})^+ = \pm ia \quad (N)^+ = N - i\frac{(2l+1)\pi}{\epsilon} \quad (13)$$

where $l \in \mathbf{Z}$ here is the same integer as that in (4). It is important to note that the above involution corresponds to the standard conjugation in contrast to the non-standard conjugation for (7).

To obtain a deeper insight into these conjugations, it is instructive to look at the representations of (1) under these conditions. Let us first consider the algebra with the involution as defined by (7). To obtain its representations, we first note that N commutes with both $\bar{a}a$ and $a\bar{a}$. As a consequence it is possible to construct a vector $|\psi_0\rangle$ which is a simultaneous eigenstate of N and $\bar{a}a$ i.e.

$$N|\psi_0\rangle = \nu_0|\psi_0\rangle \quad \bar{a}a|\psi_0\rangle = \lambda_0|\psi_0\rangle \quad (14)$$

where ν_0 and λ_0 are the corresponding eigenvalues. Note that these values are real since both the operators N and $\bar{a}a$ are hermitian. Now, from $|\psi_0\rangle$ one can construct other eigenstates of N by setting

$$|\psi_n\rangle = \bar{a}^n|\psi_0\rangle \quad |\psi_{-n}\rangle = a^n|\psi_0\rangle \quad (15)$$

for a positive integer n . It is easy to verify that

$$N|\psi_{\pm n}\rangle = (\nu_0 \pm n)|\psi_{\pm n}\rangle. \quad (16)$$

Moreover \bar{a} and a act as raising and lowering operators on the states $|\psi_n\rangle$. Indeed, to see this, consider the operator identities

$$a\bar{a}^n = \bar{a}^n a + [n]_{q^{\frac{1}{2}}} \frac{q^{(N-\frac{n}{2}+1)} + q^{-(N-\frac{n}{2}+1)}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \bar{a}^{n-1} \quad (17a)$$

$$\bar{a}a^n = a^n \bar{a} - [n]_{q^{\frac{1}{2}}} \frac{q^{(N+\frac{n}{2})} + q^{-(N+\frac{n}{2})}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} a^{n-1} \quad (17b)$$

which can be shown by induction. Then from (15) and the above relations, we have for $n > 0$,

$$\bar{a}|\psi_n\rangle = |\psi_{n+1}\rangle \quad a|\psi_n\rangle = \lambda_n|\psi_{n-1}\rangle \quad (18a)$$

$$\bar{a}|\psi_{-n}\rangle = \lambda_{-n+1}|\psi_{-n+1}\rangle \quad a|\psi_{-n}\rangle = |\psi_{-n-1}\rangle \quad (18b)$$

[†] In (12) above, it is implicit that η means $\eta\mathbf{l}$.

where

$$\lambda_n = \lambda_0 + [n]_{q^{\frac{1}{2}}} \frac{q^{\nu_0 + \frac{n}{2}} + q^{-\nu_0 - \frac{n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}. \quad (18c)$$

It is clear from above that the q -oscillator algebra admits an infinite tower of states, which is a standard feature of the usual oscillator algebra. Let us now examine the feasibility of truncating these states. Here as in the trivial case ($q = 1$), we can set $a|\psi_0\rangle = 0$ without any obstruction. This will mean that $\lambda_0 = 0$ and $|\psi_{-n}\rangle = 0$ for $n > 0$. Next let us see whether, it is possible to set $\bar{a}|\psi_k\rangle = 0$ for some state $|\psi_k\rangle$ with integer $k \geq 0$. The existence of such a state will essentially limit the non-trivial states to a finite number. Proceeding along this, we first consider the casmir

$$\mathcal{C}_2 = \bar{a}a - [N]_q = a\bar{a} - [N+1]_q \quad (19)$$

associated with algebra (1) and assume that $|\psi_k\rangle$ does exist. Then by applying both forms of \mathcal{C}_2 to $|\psi_k\rangle$ and remembering that $|q| = 1$ with $q = e^{i\epsilon}$ we have

$$\sin(\epsilon(\nu_0 + k + 1)) = \sin(\epsilon\nu_0) \quad (20a)$$

or equivalently to

$$\cos(\epsilon(2\nu_0 + k + 1)/2) \sin(\epsilon(k + 1)/2) = 0. \quad (20b)$$

Now the sine term is non zero for $k \geq 0$ for arbitrary ϵ while the cosine term can vanish if

$$\epsilon(2\nu_0 + k + 1) = (2l + 1)\pi \quad l \in \mathbf{Z} \quad (21)$$

holds. Thus if ν_0 , which at this stage is still arbitrary, is suitably chosen such that the above condition is satisfied for some k , then it is possible to have a finite representation. Implementing this, we have from (21)

$$\nu_0 = \frac{-k - 1}{2} + \frac{(2l + 1)\pi}{2\epsilon}. \quad (22)$$

As a consequence, we have[†]

$$\bar{a}|\psi_{k,n}\rangle = |\psi_{k,n+1}\rangle \quad a|\psi_{k,n}\rangle = \lambda_n|\psi_{k,n-1}\rangle \quad (23a)$$

with the finiteness in the number of states controlled by

$$a|\psi_{k,0}\rangle = 0 \quad \bar{a}|\psi_{k,k}\rangle = 0. \quad (23b)$$

[†] Here we have included the index k in the state to reflect the dependence of the representation on this parameter.

For ν_0 assuming the above value, λ_n now reads as

$$\lambda_n = (-1)^l \tan(\epsilon/2) [n]_{q^{1/2}} [k+1-n]_{q^{1/2}}. \quad (24)$$

It is important to note that the eigenstates are, as yet, not normalised. To do so, we first compute the norm:

$$\begin{aligned} \langle \psi_{k,n} | \psi_{k,n} \rangle &= \langle \psi_{k,n} | \bar{a} | \psi_{k,n-1} \rangle \\ &= \bar{\lambda}_n \langle \psi_{k,n-1} | \psi_{k,n-1} \rangle \\ &= \left(\prod_{m=1}^n \lambda_m \right) \langle \psi_{k,0} | \psi_{k,0} \rangle \end{aligned} \quad (25)$$

where we have replaced $\bar{\lambda}_m$ by λ_m since it is real. Now to ensure that this be positive definite, we set $\langle \psi_{k,0} | \psi_{k,0} \rangle = 1$ and require that each λ_m term in the product be positive. Here the sign is dictated by the term $(-1)^l \tan(\epsilon/2)$ (see (24)) and can be made positive if we choose l to be such that $(-1)^l \tan(\epsilon/2) > 0$. For instance, if $0 < \epsilon < \pi/2$ then l can be chosen to be even.

With this we can write the normalized states as

$$|k, n\rangle = \frac{|\psi_{k,n}\rangle}{\langle \psi_{k,n} | \psi_{k,n} \rangle^{1/2}} \quad 0 \leq n \leq k \quad (26)$$

with

$$\bar{a}|k, n\rangle = ((-1)^l \tan(\epsilon/2))^{1/2} ([n+1]_{q^{1/2}} [k-n]_{q^{1/2}})^{1/2} |k, n+1\rangle \quad (27a)$$

$$a|k, n\rangle = ((-1)^l \tan(\epsilon/2))^{1/2} ([n]_{q^{1/2}} [k+1-n]_{q^{1/2}})^{1/2} |k, n-1\rangle \quad (27b)$$

$$N|k, n\rangle = \left(\frac{-k-1}{2} + \frac{(2l+1)\pi}{2\epsilon} + n \right) |k, n\rangle. \quad (27c)$$

Here it should be noted that the representations are valid only for $\epsilon \neq 0$ or equivalently $q \neq 1$. This is important, as one would otherwise be led to an undeformed oscillator algebra with finite dimensional representations. This clearly would have been a paradox as it is well known that there are no such representations for the usual oscillator algebra.

On further examination, an even more interesting fact surfaces. To see this, we first relabel the parameters k and n by $k = 2j$ and $n = j + m$. Then by replacing $|k, n\rangle \rightarrow |j, m\rangle$, we have

$$((-1)^l \cot(\epsilon/2))^{1/2} \bar{a}|j, m\rangle = ([j-m]_{q^{1/2}} [j+m+1]_{q^{1/2}})^{1/2} |j, m+1\rangle \quad (28a)$$

$$((-1)^l \cot(\epsilon/2))^{1/2} a|j, m\rangle = ([j+m]_{q^{1/2}} [j-m+1]_{q^{1/2}})^{1/2} |j, m-1\rangle \quad (28b)$$

$$\left(N + \frac{1}{2} - \frac{(2l+1)\pi}{2\epsilon} \right) |j, m\rangle = m|j, m\rangle \quad (28c)$$

where some terms have been transposed to the left hand side and m and j now assume the values $-j \leq m \leq j$, $j = 0, 1/2, 1, 3/2, \dots$. What is surprising here is that the vectors on the right hand side are precisely those of $\text{su}_{q^{1/2}}(2)$. This immediately suggests that

$$J_+ = ((-1)^l \cot(\epsilon/2))^{1/2} \bar{a} \quad (29a)$$

$$J_- = ((-1)^l \cot(\epsilon/2))^{1/2} a \quad (29b)$$

$$J_0 = N + \frac{1}{2} - \frac{(2l+1)\pi}{2\epsilon} \quad (29c)$$

which essentially establishes a linear relationship between the generators of $\mathcal{H}_q(1)$ and those of $\text{su}_{q^{1/2}}(2)$. It should be noted that the relationship not only breaks down at $\epsilon = 0$, but also at the $\epsilon = p\pi$ and $\epsilon = (2p+1)\pi/2$ where $p \in \mathbf{Z}$. Now the identification has been made through the use of a representation. One might be prompted to ask whether they hold as operator identities. To see if they do, all one needs to show is that the commutation relations of one algebra reproduces the other under (29). One can easily verify that this is indeed the case. In addition the Hopf structure of $\mathcal{H}_q(1)$ also induces through the above relationship, the Hopf structure of $\text{su}_{q^{1/2}}(2)$. This implies that for $|q| = 1$, modulo values for which (29) is ill-defined, the Hopf $*$ -algebra of $\mathcal{H}_q(1)$ with non-standard conjugation is equivalent to $\text{su}_{q^{1/2}}(2)$.

Next, let us consider the representations of $\mathcal{H}_q(1)$ under the involution provided by (13).[†] As in the previous case we start with representations (18) with $\lambda_0 = 0$. Then under the standard prescription

$$\langle \psi_n | N^+ | \psi_n \rangle = \overline{\langle \psi_n | N | \psi_n \rangle} \quad (30)$$

one is led to the condition

$$\nu_0 - \bar{\nu}_0 = i \frac{(2l+1)\pi}{\epsilon} \quad (31)$$

which implies that

$$\nu_0 = \mu_0 + i \frac{(2l+1)\pi}{2\epsilon} \quad \mu_0 \in \mathbf{R}. \quad (32)$$

Let us see whether it is possible to truncate the states. The existence of $|\psi_k\rangle$ with $\bar{a}|\psi_k\rangle = 0$ now implies that

$$\cosh(\epsilon(\mu_0 + k + 1)) = \cosh(\epsilon\mu_0) \quad (33)$$

[†] In the following we choose the involution with the negative sign in (13).

which is obtained in a manner analogous to (20a). This in turn suggests that

$$\mu_0 + k + 1 = \pm u_0 \quad (34)$$

which provides us with a possible solution $k = -2\mu_0 - 1$. Since μ_0 is arbitrary we can assign to it, the values $-1/2, -1, -3/2, \dots$ which sets k to $0, 1, 2, \dots$ as required. Thus if we have

$$\nu_0 = \frac{-k-1}{2} + i \frac{(2l+1)\pi}{2\epsilon} \quad l \in \mathbf{Z} \quad (35)$$

then a truncation is certainly possible. Under this condition λ_n can be expressed as

$$\lambda_n = i(-1)^{l+1} \tanh(\epsilon/2) [n]_{q^{1/2}} [k+1-n]_{q^{1/2}} \quad (36)$$

with $0 \leq n \leq k$ and $k = 0, 1, 2, \dots$. The norms can be computed as in (25) and are given by

$$\langle \psi_n | \psi_n \rangle = \prod_{m=1}^n ((-1)^{l+1} \tanh(\epsilon/2) [m]_{q^{1/2}} [k+1-m]_{q^{1/2}}). \quad (37)$$

To ensure positive definiteness we choose l to be even for $\epsilon < 0$ and odd for $\epsilon > 0$. Then using the form of (26) for the normalized states we have

$$\bar{a}|k, n \rangle = ((-1)^{l+1} \tanh(\epsilon/2))^{1/2} ([n+1]_{q^{1/2}} [k-n]_{q^{1/2}})^{1/2} |k, n+1 \rangle \quad (38a)$$

$$a|k, n \rangle = i((-1)^{l+1} \tanh(\epsilon/2))^{1/2} ([n]_{q^{1/2}} [k+1-n]_{q^{1/2}})^{1/2} |k, n-1 \rangle \quad (38b)$$

$$N|k, n \rangle = \left(\frac{-k-1}{2} + i \frac{(2l+1)\pi}{2\epsilon} + n \right) |k, n \rangle. \quad (38c)$$

It should be noted that the above representations also break down at $\epsilon = 0$. Now these are very similar to (27) with the exception of some complex factors in the terms. In view of this similarity, we are again prompted to make the substitution $k = 2j$ and $n = j + m$ which yields

$$((-1)^{l+1} \coth(\epsilon/2))^{1/2} \bar{a}|j, m \rangle = ([j-m]_{q^{1/2}} [j+m+1]_{q^{1/2}})^{1/2} |j, m+1 \rangle \quad (39a)$$

$$-i((-1)^{l+1} \coth(\epsilon/2))^{1/2} a|j, m \rangle = ([j+m]_{q^{1/2}} [j-m+1]_{q^{1/2}})^{1/2} |j, m-1 \rangle \quad (39b)$$

$$\left(N + \frac{1}{2} - i \frac{(2l+1)\pi}{2\epsilon} \right) |j, m \rangle = m|j, m \rangle \quad (39c)$$

with the implication that

$$J_+ = ((-1)^{l+1} \coth(\epsilon/2))^{1/2} \bar{a} \quad (40a)$$

$$J_- = -i((-1)^{l+1} \coth(\epsilon/2))^{1/2} a \quad (40b)$$

$$J_0 = N + \frac{1}{2} - i \frac{(2l+1)\pi}{2\epsilon}. \quad (40c)$$

After affirming that the relationship holds at the algebraic level, we again conclude that $\mathcal{H}_q(1)$ together with the conjugation (13) is equivalent to $\text{su}_{q^{1/2}}(2)$ for $\epsilon \neq 0$ but now with the standard conjugation.

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